

# MODULAR FORMS OF ARBITRARY EVEN WEIGHT WITH NO EXCEPTIONAL PRIMES

JEFFREY HATLEY

**ABSTRACT.** A result of Dieulefait-Wiese proves the existence of modular eigenforms of weight 2 for which the image of every associated residual Galois representation is as large as possible. We generalize this result to eigenforms of general even weight  $k \geq 2$ .

## 1. INTRODUCTION

The purpose of this note is to provide a modest generalization of a theorem of Dieulefait-Wiese. Before stating the result, we briefly recall some terminology and notation.

Let  $f = \sum a_n q^n \in S_k(\Gamma_0(N))$  be a normalized cuspidal modular eigenform (henceforth simply called an “eigenform”) of weight  $k \geq 2$  and level  $\Gamma_0(N)$  for some integer  $N \geq 1$ . Let  $G_{\mathbf{Q}}$  denote the absolute Galois group  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ . The Fourier coefficients  $\{a_i\}$  generate a number field  $K_f$ . Let  $\mathcal{O}_f$  be the ring of integers of  $K_f$ , let  $\lambda$  be a maximal ideal in  $\mathcal{O}_f$  with residue characteristic  $\ell$ , and write  $\mathbf{F}_{\lambda}$  for the extension of  $\mathbf{F}_{\ell}$  generated by  $\{a_i \bmod \lambda\}$ , the residues of the Hecke eigenvalues. By work of Deligne, there is a Galois representation

$$\rho_{f,\lambda} : G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathcal{O}_{f,\lambda})$$

as well as an associated semisimple residual representation

$$\bar{\rho}_{f,\lambda} : G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_{\lambda}).$$

These representations are unramified outside the primes dividing  $N\ell\infty$ , and  $\bar{\rho}_{f,\lambda}$  is absolutely irreducible for almost all primes  $\lambda$ . Upon composing  $\bar{\rho}_{f,\lambda}$  with the natural projection  $\text{GL}_2(\mathbf{F}_{\lambda}) \rightarrow \text{PGL}_2(\mathbf{F}_{\lambda})$ , we obtain the projective representation

$$\bar{\rho}_{f,\lambda}^{\text{proj}} : G_{\mathbf{Q}} \rightarrow \text{PGL}_2(\mathbf{F}_{\lambda}).$$

By a result of Ribet [11, Theorem 3.1], if  $f$  does not have complex multiplication (CM), then the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is “as large as possible” for all but finitely many primes  $\lambda$ . More precisely, for almost all  $\lambda$ , the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is either  $\text{PGL}_2(\mathbf{F}_{\lambda})$  or  $\text{PSL}_2(\mathbf{F}_{\lambda})$  (see also [5, Corollary 3.2]). In Section 1.1 we briefly discuss the history of such results.

**Definition 1.** A maximal ideal  $\lambda$  of  $\mathcal{O}_f$  is called *exceptional* if the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is not  $\text{PGL}_2(\mathbf{F}_{\lambda})$  or  $\text{PSL}_2(\mathbf{F}_{\lambda})$ . We may also say that  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is exceptional.

*Remark 1.* Recall that by Dickson’s classification, if  $\bar{\rho}_{f,\lambda}$  is both irreducible and exceptional, then the image must be either dihedral or isomorphic to  $A_4$ ,  $S_4$ , or  $A_5$ .

Thus Ribet’s theorem states that if  $f$  does not have CM, then it has only finitely many exceptional primes. The following theorem was proved by Dieulefait-Wiese.

**Theorem 1.** [5, Theorem 6.2] *There exist eigenforms  $(f_n)_{n \in \mathbf{N}}$  of weight 2 such that*

- (1) *for all  $n$  the eigenform  $f_n$  has no exceptional primes, and*

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- (2) for a fixed prime  $\ell$ , the size of the image of  $\bar{\rho}_{f_n, \lambda_n}$  for  $\lambda_n \triangleleft \mathcal{O}_{f_n}$  is unbounded for running  $n$ .

*Remark 2.* The eigenforms  $f_n$  in Theorem 1 have some additional technical properties. First, they do not have CM, which is a necessary condition. Second, they have no nontrivial inner twists; this is important for their application to the Inverse Galois problem in [5]. While the modular forms which we construct in Theorem 2 also enjoy these properties, we will not mention them for the sake of brevity and ease of exposition.

In this paper, we modify the arguments of [5] to obtain a version of Theorem 1 for eigenforms of general even weight  $k \geq 2$ . The main result of this paper is the following.

**Theorem 2.** *Let  $k \geq 2$  be an even integer. There exist eigenforms  $(f_n)_{n \in \mathbf{N}}$  of weight  $k$  such that*

- (1) *for all  $n$  the eigenform  $f_n$  has no exceptional primes, and*
- (2) *for a fixed prime  $\ell$ , the size of the image of  $\bar{\rho}_{f_n, \lambda_n}$  for  $\lambda_n \triangleleft \mathcal{O}_{f_n}$  is unbounded for running  $n$ .*

*Remark 3.* If  $f$  is a weight 2 eigenform with trivial nebentype whose coefficients are all rational, then by the Eichler-Shimura construction, there is an elliptic curve  $E/\mathbf{Q}$  associated to  $f$ . In [3], Daniels constructed an explicit infinite family of elliptic curves over  $\mathbf{Q}$  whose adelic Galois representations have maximal image; in particular, they have no exceptional primes. In fact, Duke and Jones have shown that, in an appropriate sense, almost all elliptic curves have no exceptional primes [4, 7].

Thus, the value of Theorem 2 is in providing modular forms which are guaranteed not to correspond to elliptic curves but which nevertheless have no exceptional primes.

**1.1. Historical Context.** Given a modular form  $f$ , one can form an adelic Galois representation

$$\rho_f : G_{\mathbf{Q}} \rightarrow \prod_{\lambda} \mathrm{GL}_2(\mathcal{O}_{f, \lambda})$$

where  $\lambda$  ranges over all maximal ideals of  $\mathcal{O}_f$ . In the special case where  $f$  corresponds to an elliptic curve  $E/\mathbf{Q}$ , this is equivalent to the “full-torsion” representation

$$\rho_E : G_{\mathbf{Q}} \rightarrow \varprojlim_n \mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z}) \simeq \mathrm{GL}_2(\hat{\mathbf{Z}}).$$

Serre showed that, assuming  $E$  does not have CM, the image of  $\rho_E$  is *open* in a subgroup of index 2 inside  $\mathrm{GL}_2(\hat{\mathbf{Z}})$  [14, Proposition 22]; this implies that  $E$  has finitely many exceptional primes. As mentioned in Remark 3, more recent results have shown that, generically, an elliptic curve has no exceptional primes [4, 7].

An analogue of Serre’s theorem has recently been proven for modular forms. Loeffler showed that the adelic Galois representation attached to an arbitrary non-CM modular form of weight  $k \geq 2$  has open image [9, Theorem 2.3.1]. This relies on older results of Ribet and Momose which proved that modular forms have finitely many exceptional primes; see for instance [11, Theorem 3.1].

Nevertheless, it can be very hard to explicitly identify the exceptional primes for any given modular form. Recent work of Billerey-Dieulefait gives explicit but complicated bounds on the exceptional primes for a modular form of weight  $k \geq 2$  and trivial nebentype [1].

## 2. PRELIMINARIES

In this section we collect some definitions and basic results which will be needed in Section 3 to prove our main result.

**2.1. Tamely dihedral representations.** The notion of *tamely dihedral representations* was first defined by Dieulefait-Wiese in [5, Section 4]; their definition was inspired by the notion of *good-dihedral primes* from [8]. We first recall some facts regarding Galois representations arising from modular forms.

Let  $f$  be an eigenform, let  $K_f$  be its coefficient field and  $\mathcal{O}_f$  its ring of integers, and let  $\lambda \mid \ell$  be a prime of  $\mathcal{O}_f$  dividing a rational prime  $\ell$ . For any rational prime  $p$ , let  $G_p$  denote a decomposition group corresponding to  $p$ . For the rest of this section, let  $p$  denote a prime different from  $\ell$ . By Grothendieck's monodromy theorem we may associate to the characteristic zero local representation

$$\rho_{f,\lambda}|_{G_p}: G_p \rightarrow \mathrm{GL}_2(\mathcal{O}_{f,\lambda})$$

a 2-dimensional Weil-Deligne representation  $\tau_p = (\tilde{\rho}, \tilde{N})$ . Here

$$\tilde{\rho}: W_{\mathbf{Q}_p} \rightarrow \mathrm{GL}_2(K_{f,\lambda})$$

is a continuous representation of the Weil group of  $\mathbf{Q}_p$  for the discrete topology on  $\mathrm{GL}_2(K_{f,\lambda})$ ,  $\tilde{N}$  is a nilpotent matrix in  $M_2(K_{f,\lambda})$ , and we have the relation

$$\tilde{\rho}\tilde{N}\tilde{\rho}^{-1} = |\cdot|^{-1} \tilde{N}$$

where  $|\cdot|$  is a particular norm map. The standard reference for these things is [15], but another very readable reference is [6].

**Definition 2.** [5, Definition 4.1] Let  $\mathbf{Q}_{p^2}$  be the unique unramified degree 2 extension of  $\mathbf{Q}_p$ . Denote by  $W_p$  and  $W_{p^2}$  the Weil groups of  $\mathbf{Q}_p$  and  $\mathbf{Q}_{p^2}$ , respectively.

A 2-dimensional Weil-Deligne representation  $\tau_p = (\tilde{\rho}, \tilde{N})$  of  $\mathbf{Q}_p$  with values in  $K_f$  is called *tamely dihedral of order  $n$*  if  $\tilde{N} = 0$  and there is a tame character  $\psi: W_{p^2} \rightarrow K_{f,\lambda}^\times$  whose restriction to the inertia group  $I_p$  (which is naturally a subgroup of  $W_{p^2}$ ) is of niveau 2 (i.e. it factors over  $\mathbf{F}_{p^2}^\times$  and not over  $\mathbf{F}_p^\times$ ) and of order  $n > 2$ , such that  $\tilde{\rho} \simeq \mathrm{Ind}_{W_{p^2}}^{W_p} \psi$ .

We say that an eigenform  $f$  is *tamely dihedral of order  $n$*  at the prime  $p$  if the Weil-Deligne representation  $\tau_p$  at  $p$  associated to  $f$  is tamely dihedral of order  $n$ .

*Remark 4.* In terms of the local Langlands correspondence,  $f$  can only be tamely dihedral at  $p$  if it is *supercuspidal* at  $p$ . Recent work of Loeffler-Weinstein [10] has made it possible to test modular forms for the property of being tamely dihedral using the LocalComponent package of [13]. Thus, in theory one can find explicit examples of the modular forms whose existence is guaranteed by Theorem 2; however, as the proof of the theorem will indicate, these modular forms are expected to have very large level, and their construction seems beyond the scope of current computing capabilities.

**Proposition 1.** *Let  $f \in S_k(N, \chi_{\mathrm{triv}})$  be a newform of odd level  $N$  and trivial nebentype such that for all  $\ell \mid N$*

- (1)  $\ell \parallel N$  or
- (2)  $\ell^2 \parallel N$  and  $f$  is tamely dihedral at  $\ell$  of order  $n_\ell > 2$  or
- (3)  $\ell^2 \mid N$  and  $\rho_{f,t}(G_\ell)$  can be conjugated to lie in the upper triangular matrices such that the elements on the diagonal all have odd order for some prime  $t \nmid \ell$ .

Let  $\{p_1, \dots, p_r\}$  be any finite set of primes.

Then for almost all primes  $p \equiv 1 \pmod{4}$  there is a set  $S$  of primes of positive density which are completely split in  $\mathbf{Q}(i, \sqrt{p_1}, \dots, \sqrt{p_r})$  such that for all  $q \in S$  there is a newform  $g \in S_k(Nq^2, \chi_{\text{triv}})$  which is tamely dihedral at  $q$  of order  $p$  and for all  $\ell \mid N$  we have

- (1)  $\ell^2 \parallel N$  and  $g$  is tamely dihedral at  $\ell$  of order  $n_\ell > 2$  or
- (2)  $\rho_{g,t}(G_\ell)$  can be conjugated to lie in the upper triangular matrices such that the elements on the diagonal all have odd order for some prime  $t \nmid \ell$ .

*Proof.* This is [5, Proposition 5.4]. □

**2.2. Local  $\ell$ -adic representations.** Let  $f = \sum a_n q^n$  be an eigenform, and let  $\lambda$  be a prime of  $\mathcal{O}_f$  lying above the rational prime  $\ell$ . Recall that  $f$  is said to be *ordinary at  $\lambda$*  if  $a_\ell \not\equiv 0 \pmod{\lambda}$ ; otherwise  $f$  is said to be *nonordinary at  $\lambda$* . Let  $G_\ell$  be a decomposition group at  $\ell$  and  $I_\ell$  its inertia group.

The following theorem is due to Deligne, Fontaine, and Edixhoven.

**Theorem 3.** *Assume  $f$  is weight  $k$  and  $\bar{\rho}_{f,\lambda}$  is irreducible.*

- (1) *If  $k \geq 2$  and  $f$  is ordinary at  $\lambda$  then*

$$\bar{\rho}_{f,\lambda} |_{I_\ell} \simeq \begin{pmatrix} \chi_\ell^{k-1} & * \\ 0 & 1 \end{pmatrix}$$

*where  $\chi_\ell$  is the (reduction of the)  $\ell$ -adic cyclotomic character.*

- (2) *If  $2 \leq k \leq \ell + 1$  and  $f$  is nonordinary at  $\lambda$  then*

$$\bar{\rho}_{f,\lambda} |_{I_\ell} \simeq \begin{pmatrix} \phi^{k-1} & 0 \\ 0 & \phi^{\ell(k-1)} \end{pmatrix}$$

*where  $\phi$  is a fundamental character of niveau 2.*

*Proof.* We refer to reader to [2, Theorem 1.2] and the remark which follows it. □

Thus, the image of inertia under  $\bar{\rho}_{f,\lambda}$  can be identified with the image of  $\chi_\ell^{k-1}$  or  $\phi^{(l-1)(k-1)}$  depending on whether  $f$  is ordinary or nonordinary at  $\lambda$ . In particular, we have the following corollary.

**Corollary 1.** *Assume  $\ell > k$ . Let  $\mathcal{I} = \bar{\rho}_{f,\lambda}^{\text{proj}}(I_\ell)$ .*

- (1) *If  $f$  is ordinary at  $\lambda$ , then  $\mathcal{I}$  is cyclic of order  $n = (\ell - 1)/\gcd(\ell - 1, k - 1) \geq 2$ . If  $\ell > 5k - 4$ , then  $n > 5$ .*
- (2) *If  $f$  is nonordinary at  $\lambda$ , then  $\mathcal{I}$  is cyclic of order  $n = (\ell + 1)/\gcd(\ell + 1, k - 1) \geq 2$ . If  $\ell > 5k - 4$ , then  $n > 5$ .*

*Proof.* This follows immediately from Theorem 3; see also [1, Lemma 1.2] □

We conclude this section with a lemma which is the crucial ingredient for generalizing from weight 2 forms to weight  $k$  forms. The first argument of this sort, for the  $k = 2$  case, goes back to Ribet (see the proof of [12, Proposition 2.2]). For higher weights, see [1, Section 3.3], which we follow closely.

Let  $\mathcal{G} = \bar{\rho}_{f,\lambda}^{\text{proj}}(G_{\mathbf{Q}})$  be the projective image of  $\bar{\rho}_{f,\lambda}$ , and suppose  $\mathcal{G}$  is dihedral. Then  $\mathcal{G}$  fits into an exact sequence of the form

$$0 \rightarrow \mathcal{Z} \rightarrow \mathcal{G} \rightarrow \{\pm 1\} \rightarrow 0$$

where  $\mathcal{Z}$  is cyclic. This corresponds to a tower of fields

$$\mathbf{Q} \subset E \subset L$$

with Galois groups

$$\text{Gal}(L/\mathbf{Q}) \simeq \mathcal{G}, \quad \text{Gal}(E/\mathbf{Q}) \simeq \{\pm 1\}, \quad \text{Gal}(L/E) \simeq \mathcal{Z}.$$

We thus obtain a quadratic character  $\epsilon : G_{\mathbf{Q}} \rightarrow \{\pm 1\}$  whose kernel cuts out  $E$ .

**Lemma 1.** *If  $\ell > 5k - 4$ , then  $\epsilon$  is unramified at  $\ell$ .*

*Proof.* By Corollary 1,  $\mathcal{I}$  is cyclic of order  $> 5$ . Since  $\mathcal{I} \subset \mathcal{G}$ , we must have  $\mathcal{I} \subset \mathcal{Z}$ . Thus  $I_{\ell}$  is contained in the kernel of  $\epsilon$ .  $\square$

### 3. MAIN RESULT

In order to prove our main theorem, we must first prove a version of [5, Proposition 6.1] for eigenforms of general weight  $k \geq 2$ , after which the proof of our theorem will follow easily.

**Proposition 2.** *Let  $p, q, t, u$  be distinct odd primes and let  $N$  be an integer which is divisible by every odd prime  $p \leq 5k - 4$ . Let  $p_1, \dots, p_m$  be the prime divisors of  $2N$ . Let  $f \in S_k(Nq^2u^2, \chi)$  be an eigenform without CM which is tamely dihedral of order  $p^r > 5$  at  $q$  and tamely dihedral of order  $t^s > 5$  at  $u$ . Assume that  $q$  and  $u$  are completely split in  $\mathbf{Q}(i, \sqrt{p_1}, \dots, \sqrt{p_m})$  and that  $(\frac{q}{u}) = (\frac{u}{q}) = 1$ .*

*Then  $f$  does not have any exceptional primes, i.e. for all maximal ideals  $\lambda$  of  $\mathcal{O}_f$ , the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is  $\text{PSL}_2(\mathbf{F}_{\lambda})$  or  $\text{PGL}_2(\mathbf{F}_{\lambda})$ .*

*Proof.* The proof is similar to the proof of [5, Proposition 6.1], which we follow closely. Let  $\lambda$  be any maximal ideal of  $\mathcal{O}_f$  and suppose it lies over the rational prime  $\ell$ . By our “tamely dihedral” hypotheses,  $\bar{\rho}_{f,\lambda}$  is irreducible, since if  $\ell \notin \{p, q\}$ , then already  $\bar{\rho}_{f,\lambda} |_{G_q}$  is irreducible, and if  $\ell \in \{p, q\}$ , then  $\ell \notin \{t, u\}$ , hence  $\bar{\rho}_{f,\lambda} |_{G_u}$  is irreducible.

Now suppose the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is a dihedral group. This means that  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is the induction of a character of a quadratic extension  $E/\mathbf{Q}$ , i.e.

$$\bar{\rho}_{f,\lambda}^{\text{proj}} \simeq \text{Ind}_E^{\mathbf{Q}}(\alpha)$$

for some character  $\alpha$  of  $\text{Gal}(\bar{\mathbf{Q}}/E)$ . By the ramification properties of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$ , we know

$$(1) \quad E \subset \mathbf{Q}(i, \sqrt{\ell}, \sqrt{q}, \sqrt{u}, \sqrt{p_1}, \dots, \sqrt{p_m}).$$

First assume that  $\ell \notin \{p, q\}$ . In this case, we have

$$\bar{\rho}_{f,\lambda}^{\text{proj}} |_{D_q} \simeq \text{Ind}_{\mathbf{Q}_{q^2}}^{\mathbf{Q}_q}(\psi) \simeq \text{Ind}_{E_q}^{\mathbf{Q}_q}(\alpha)$$

where  $\mathfrak{q}$  is a prime in  $\mathcal{O}_E$  lying over  $q$  and where  $\psi$  is a niveau 2 character of order  $p^r$ . This implies that  $q$  is inert in  $E$ , but by assumption  $q$  is totally split in  $\mathbf{Q}(i, \sqrt{u}, \sqrt{p_1}, \dots, \sqrt{p_m})$ , so from (1) we deduce that

$$\ell \notin \{u, p_1, \dots, p_m\}.$$

In particular, we see that  $\ell \nmid 2Nu$ , so by our choice of  $N$ , we conclude that  $\ell > 5k - 4$ . Thus by Lemma 1 our quadratic field  $E$  cannot ramify at  $\ell$ , so we can refine (1) to

$$E \subset \mathbf{Q}(i, \sqrt{q}, \sqrt{u}, \sqrt{p_1}, \dots, \sqrt{p_m}),$$

with  $E$  totally split in the latter. But now the fact that  $q$  is inert in  $E$  implies that  $E = \mathbf{Q}$  rather than a quadratic extension, and this contradiction implies that  $\ell \in \{p, q\}$  and in particular  $\ell \notin \{t, u\}$ . Upon exchanging the roles  $q \leftrightarrow u$ ,  $p \leftrightarrow t$ , and  $r \leftrightarrow s$ , running this argument again leads to a contradiction, hence the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is not dihedral.

If  $\lambda$  is exceptional and the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  is not dihedral, then by Dickson's classification, the only other possibilities for the image are  $A_4$ ,  $S_4$ , and  $A_5$ . But the image of  $\bar{\rho}_{f,\lambda}^{\text{proj}}$  contains an element of order  $> 5$  by Corollary 1, so none of these are possible.  $\square$

We may now prove our main theorem. The proof is essentially the same as the proof of [5, Theorem 6.2].

*Proof of Theorem 2.* Start with some newform  $f_1 \in S_k(\Gamma_0(N))$  for  $N$  of squarefree level. Note that modular forms of level  $\Gamma_0(N)$  never have CM when  $N$  is squarefree. Let  $p_1, \dots, p_m$  be the prime divisors of  $6N$ .

Let  $B_1 > 0$  be any bound. Take  $p$  to be any prime bigger than  $B$  provided by Proposition 1 applied to  $f$  and the set  $\{p_1, \dots, p_m\}$ . We thus obtain an eigenform  $g \in S_k(\Gamma_0(Nq^2))$  which is tamely dihedral at  $q$  of order  $p$  for some prime  $q$ . Now apply Proposition 1 to the form  $g$  and the set  $\{q, p_1, \dots, p_m\}$  to obtain a prime  $t > B$  different from  $p$  and an eigenform  $h \in S_k(\Gamma_0(Nq^2u^2))$  which is tamely dihedral at  $u$  of order  $t$  for some prime  $u$ . By Proposition 2,  $h$  does not have any exceptional primes.

Thus we take  $f_2 = h$  and take a new bound  $B_2 > B_1$ . Inductively we obtain a family  $(f_n)_{n \in \mathbb{N}}$  and the image of inertia grows without bound in this family.  $\square$

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